Sunspots

Week 9, PHYS 777
Sunspots have lower temperature and stronger magnetic fields than the surrounding photosphere. They can be simple or complicated. In the inner part, the umbra, there exist umbral dots and/or light bridges. The outer portion is called the penumbra.

White-light (520 ± 3 nm) speckle reconstruction of a sunspot (NOAA 8673) observed on 27 August 1999 with the 65-cm vacuum reflector at BBSO. The field of view is 76 arcsec x 76 arcsec.
Magnetoconvection

Consider only the balance between the buoyancy force \((g \Delta \rho)\) and the viscous force \((\pi^2 \rho \nu v/d^2)\). I few label \(\alpha = \Delta \rho / (\Delta T \rho)\) the volume expansion coefficient, the force balance may be written as \(\rho g \alpha \Delta T \approx \pi^2 \rho \nu v/d^2\).

Now, if the parcel is not thermally isolated by an inward heat leak with diffusivity \(\kappa\) the condition for stability becomes

\[
d / v = d^2 / (\pi^2 \kappa).
\]

This means that a parcel may move a distance \(d\) before the perturbation is smoothed out by thermal diffusion (in a timescale \(d^2 / (\pi^2 \kappa)\)). We then have that

\[
Ra \equiv (g \alpha \Delta T d^3) / (\kappa \nu)
\]

The magnetic field tries to stabilize convective motion through its tension.
For a wave-number $k$, and a vertical displacement $\xi$ the magnetic tension $B_0/\mu^2$ produces a restoring force $k^2 \xi B_0^2/\mu$ which is balanced by the buoyancy force $g(\rho_0\alpha \xi \Delta T /d)$.

The instability condition then becomes

$$g \rho_0 \alpha \Delta T /d > k^2 B_0^2/\mu.$$
Furthermore, if the thermal diffusivity $\kappa$ and magnetic diffusivity $\eta$ are considered, we have:

If $\kappa < \eta$, we have the \textit{leak instability} condition

$$\frac{(\rho_0 g \alpha \Delta T \eta)}{d} > \frac{(k^2 B_o^2 \kappa)}{\mu}$$

If $\kappa > \eta$, we have overstability

$$\frac{(\rho_0 g \alpha \Delta T \kappa)}{d} > \frac{(k^2 B_o^2 \eta)}{\mu}$$
Linear Stability Analysis

\[ \rho(Dv/Dt) = -\nabla \rho + j \times B + \rho v \nabla^2 v - \rho g \hat{z} \quad (\hat{z} \text{ is unit vector in } z \text{ dir}) \]

\[ \partial B/\partial t = \nabla \times (v \times B) + \eta \nabla^2 B \]

\[ DT/Dt = \kappa \nabla^2 T \]

where \( \nabla \cdot v = \nabla \cdot B = 0 \)

Departures from equilibrium are explored using

\( B = B_o + B_1, \, v = v_1, \, T = T_o(z) + T_1, \) and \( \rho = \rho_o(1+\alpha T_1) \)

After eliminating all but one variable, say \( v_{1z} \), the equations reduce to

\[ \left( \partial/\partial t - \kappa \nabla^2 \right) \left( \partial/\partial t - \eta \nabla^2 \right) \left( \partial/\partial t - \nu \nabla^2 \right) \nabla^2 v_{1z} = \]

\[ \left( (B_o \cdot \nabla)^2/ \mu \rho_o \right) \left( \partial/\partial t - \kappa \nabla^2 \right) \nabla^2 v_{1z} + \]

\[ (g\alpha \Delta T/d) \left( \partial/\partial t - \eta \nabla^2 \right) \left( \partial^2 v_{1z}/\partial x^2 + \partial^2 v_{1z}/\partial y^2 \right) \]
We now assume a solution of the form

\[ V_{1z} \sim \exp(\omega t) \ast \exp(i[k_x x + k_y y]) \ast \sin(k_z z), \text{ with } k_z = \frac{\pi}{d} \text{ to satisfy boundary conditions in } z \text{ direction.} \]

The previous equation then reduces to

\[
(\omega + \kappa k^2)(\omega + \eta k^2)(\omega + \nu k^2) k^2 =
\]

\[-((B_0 \cdot k)^2/\mu \rho_o)(\omega + \kappa k^2) k^2 + (g \alpha \Delta T/d)(\omega + \eta k^2)(k_x^2 + k_y^2)\]
Now consider a horizontal magnetic field \((B_0 = B_0 \hat{x})\) and neglect dissipative effects \((\nu = \eta = \kappa = 0)\). The dispersion relation is then

\[
\omega^2 = - \left( \frac{B_0^2}{\mu \rho_0} \right) k_x^2 + \left( \frac{g\alpha \Delta T}{\alpha} \right) \left( \frac{k_x^2 + k_y^2}{k^2} \right)
\]

If \(k_x = 0\) all modes are unstable. If \(k_x \neq 0\), stable condition

\[
\left( \frac{B_0^2}{\mu \rho_0} \right) > \left( \frac{g\alpha \Delta T}{d} \right) \left( \frac{k_x^2 + k_y^2}{k^2 k_x^2} \right)
\]
For vertical fields \((\mathbf{B}_o = B_0 \hat{z})\) when \(\nu, \eta,\) and \(\kappa\) are considered the condition for stability becomes more complicated. Set \(\omega = 0\)

\[
Ra \left( d^2 k^2 - \pi^2 \right) = \pi^2 H^2 a^2 d^2 k^2 + d^6 k^6,
\]

where \(H = B_o d / (\nu \eta \rho \mu)^{1/2}\) is the Hartmann number

\(H \gtrsim 1\) for Sun so \(Ra^* \approx \pi^2 H^2\)

For stability (growing oscillation)

\[
\omega = \omega_r + i \omega_i \quad \omega_r > 0
\]

\[
Ra \left( d^2 k^2 - \pi^2 \right) = \pi^2 H^2 a^2 d^2 k^2 \left( \eta (\nu + \eta) \right) / \left( \kappa (\nu + \kappa) \right)
\]

\[
+ d^6 k^6 (\eta + \nu) (\eta + \kappa) / (\kappa \nu)
\]

For \(H \gtrsim 1\) \(Ra \approx \pi^2 H^2 \eta^2 / \kappa^2\)
Magnetic flux expulsion and concentration

If $B$ is weak $B^2/\mu = (1/2)\rho v^2$ and the magnetic Reynolds number $Rm = (\ell_o v_o)/\eta ? 1$. Magnetic Fields are just carried by flows until the magnetic energy is comparable to the kinetic energy.

In Parker’s simple kinematic concentration model consider an incompressible flow $\mathbf{v} = v_o \sin (kx) \hat{x} + v_o k z \cos (kx) \hat{z}$ on a uniform, vertical ($B_0 \hat{z}$) field.

The $z$ component of the induction equation becomes

$$\frac{\partial B_z}{\partial t} = - \frac{\partial}{\partial x} (v_o \sin k x B_z)$$

$$B_z = B_0 \exp(-kv_o t) / [\cos^2 (k x) + \sin^2 (k x) \exp(-2kv_o t)]$$

At $x = 0$, upwelling motion $B_z \approx B_0 \exp(-kv_o t)$, and field is dispersed

At $x = \pi/k$ we have a downdraft part $B_z \approx B_0 \exp(kv_o t)$
In a more precise work by Galloway and Weiss (1981) flow is described by a stream function

\[ \psi = \left( \frac{UL}{\pi} \right) \cos(\pi x/L) \cos(\pi z/L) \]

Where \( U \) and \( L \) are the characteristic speed and length. For supergranules \( t = 20 \) hours (?) and \( L = 30,000 \) km, \( U = L/t \)

Induction equation for the time development of an initially uniform vertical magnetic field:

\[ \frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B \]

\[ Rm = \frac{UL}{\eta} , \tau = \frac{L}{U} \text{ (diffusion time)} , \]

\[ \tau_d = \frac{\ell^2}{\eta} , \text{ where } \ell = \left( \frac{B_0}{B} \right) L \]

After 6 \( L/U \) nearly all the fluxes are concentrated at the boundary (see next slide).
When \( \ell \) is small diffusion becomes effective. Max concentration

\[
B_m = \left( \frac{L}{\ell} \right) B_0 = Rm^{1/2} B_0
\]

in 2-D \( B_m = Rm \)

Further consideration:

Kinematic solution has

Mag. energy = gas energy

\[
B_0^2 / 2\mu = \frac{1}{\rho} U^2,
\]

\[
B_m = B_0
\]

It is a few hundred gauss in the photosphere, and \( 10^4 \) gauss at the base of the convection zone.
Magnetic Buoyancy (Emergence of sunspots)

Isothermal flux tubes must rise since they are lighter than their surroundings.

\[
\left( \frac{k_B T \rho_e}{m} \right) = \left( \frac{k_B T \rho_i}{m} \right) + \left( \frac{B_i^2}{\mu L} \right), \text{ so } \rho_e > \rho_i.
\]

Buoyancy force per unit volume = \((\rho_e - \rho_i)g\).

However, a curved tube can provide a tension force \(\rho_e / \mu L = F_T\) to balance with buoyancy. Condition for buoyancy would then be \((\rho_e - \rho_i)g > \frac{B_i^2}{2\mu}, L > 2k_B T / mg = 2\Lambda\)
A tube that is longer than twice the local scale height will rise. The buoyancy effect can be estimated from the size of the density deficit

\[ \frac{\rho_e - \rho_i}{\rho_e} = \frac{B_i^2 m}{2 \mu \rho_o k T} \]

<table>
<thead>
<tr>
<th>z (km)</th>
<th>( \rho ) (kg/m(^2))</th>
<th>T(K)</th>
<th>B(G)</th>
<th>( \frac{\rho_e - \rho_i}{\rho_e} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20,000</td>
<td>0.25</td>
<td>2.5x10(^5)</td>
<td>1000</td>
<td>10(^{-5})</td>
</tr>
<tr>
<td>-1000</td>
<td>0.8x10(^{-5})</td>
<td>1.5x10(^4)</td>
<td>1000</td>
<td>0.004</td>
</tr>
</tbody>
</table>

Buoyancy force is much stronger in the upper convective zone.
Magnetic Buoyancy Instability

In general, if B-fields fall off too fast instability occurs in the uniform fields.

The condition is \( \frac{d}{dz}(\frac{B_o}{\rho_o}) < 0 \). If normal buoyancy is considered the instability condition is

\[
\left(\frac{\rho_o}{\Lambda B_o}\right) \frac{d}{dz}(\frac{B_o}{\rho_o}) < -\gamma \frac{N^2}{V_A^2}, \text{ N is Brunt frequency.}
\]

Consider different wave numbers.

\[
\left(\frac{1}{\Lambda B_o}\right) \frac{dB_o}{dz} < -K^2 - \gamma \frac{N^2}{V_A^2}
\]
Consider rapid rotation: \((\Lambda \rho_o / \gamma B_o) \cdot d/dz(B_o / \rho_o) < -3\sqrt{3} \pi^2 / C_A\)
where \(C_A = V_A^2 / (2 \Omega \eta)\). Rotation makes instability grow slowly.

**Rise of flux tubes in the Sun**

New flux emergence

Time scale for rising

(Alfvén time) \(\tau = \Lambda / V_A\)

This gives 2 months for

depth of 20,000 km. Actually slowed down by: (i) drag, (ii) rotation \((\tau = 4\Omega \Lambda^2 / V_A^2)\), (iii) magnetic diffusion
Cooling of Sunspots (two explanations)

(1) Inhibition of convection near solar surface. Convection carries energy, magnetic fields can inhibit convection and block heat to then cool sunspots. Difficulty: How does the blocked heat escape? Possible solution: faculae around sunspot.

(2) Convective overstability converts to Alfvén wave and dissipates. Parker estimated the flux of waves to be $2.5 \times 10^7 \text{ W/m}^2$ for $V_A = 15 \text{ km/s}$ and $B = 3000 \text{ G}$

Reflection of such a wave will cause a reduction of upward speed and a downdraft of 1 to 2 K.
Sunspot structure

Magnetohydrostatic equilibrium. A simple model assumes

$B = B(R) \hat{z}$, $B(0) = B_{\text{max}}$, $B(\infty) = 0$ (fig 8.9 above) Pressure balance in the vertical direction gives $p(R,z) + B^2(R)/2\mu = p_e(z)$ and $\partial p/\partial z = -\rho g$, which at large distances is $dp_e/dz = -\rho_e g$. On the axis we have $p_i(z) + B_i^2(R)/2\mu = p_e(z)$, $dp_i/dz = -p_i g$ so that $dp_i/dz = dp_e/dz$ and $\rho_i = \rho_e$. We also have that
\( T_i(z)/T_e(z) = 1 - \left( \frac{B_i^2}{2\mu_0 \rho_e(z)} \right) \), which means that vertical fields have no density effect but produce a pressure deficit and hence a temperature deficit to maintain a horizontal pressure balance. A more realistic model is shown in figure 8.10.
Equilibrium in two regions:

\[ \nabla (p_i + B_i^2/2 \mu) = \rho_i g + (B_i \cdot \nabla) B_i/\mu \quad \text{and} \quad \nabla p_e = \rho_e g \]

with boundary condition \( p_i + B_i^2/2 \mu = p_e \)

If \( B_i \) is potential it may be written

\[ B_i = (1/R)(- \partial \psi / \partial z, 0, \partial \psi / \partial R) \]

in cylindrical polars, where \( \psi(R,z) \) satisfies

\[ \partial^2 \psi / \partial R^2 - (1/R) \partial \psi / \partial R + \partial^2 \psi / \partial z^2 = 0 \quad (\nabla \cdot B = 0) \]

as long as the total pressure is continuous ( \( B_i^2/2 \mu = p_e(z) \)).

As \( z \to \infty \) field is horizontal \( B_R \approx B_i \approx F / 2\pi a^2 \)

As \( z \to -\infty \) field becomes vertical \( B_z \approx B_i \approx F / \pi a^2 \)

A simple (vacuum) solution is a Bessel function with field

\[ B_R = A \kappa J_1(kR) \exp(-kz) \]
Dipole model: \( B_R = 3\psi^3\cos(\theta) / A^2\sin^3(\theta) \)

Perforated current sheet model:

\[ B_R = \frac{\tanh(u)\sin(u)}{[\cosh^2(u) - \sin^2(v)]} \]

where \((u, v, \phi)\) are oblate spheroidal coordinates such that \( R = \cosh(u)\sin(v) \) and \( z = \sinh(u)\cos(v) \)

Mean model (slender flux tube):

\[ B_1(z) \approx (2 \mu p_c(z))^{1/2} \]

Parker’s spaghetti model: (fig 8.11 to the right.)
Sunspot instability

Possible instabilities: interchange, flute, and Rayleigh-Taylor
3 factors are considered: magnetic, pressure, and gravitational

Using the energy principle:

\[ \delta W_s = \int_s (n \cdot \xi)^2 n \cdot [\nabla (p_i + B_i^2/2 \mu) - \nabla p_e] \, dS \]

condition for stability is

\[ n \cdot [\nabla (p_i + B_i^2/2 \mu) - \nabla p_e] > 0 \]

For an axisymmetric field (eq 8.54 & 8.56) the stability condition reduces to \( dB_R^2/dz < 0 \). Parker found that for uniform T, the stability condition is \( R_c \sin(\chi) > 2\Lambda_e \), where \( \Lambda_e = p_e / \rho_e g \), \( R_c \) is radius of curvature of a field line in S, and \( \chi \) is the inclination of S to the vertical.
Slow decay phase (fig 8.12, right)

Ohmic diffusion \( \tau_d = \ell^2/\eta \). For a scale length of 3000 km and \( \eta = 300 \text{ m}^2/\text{s} \) \( \tau_d = 1000 \text{ yrs} \) Too long. The dominant factor is the eddy diffusivity \( \eta \). \( \eta = 2 \times 10^7 \text{ m}^2/\text{s} \)

Simple diffusion equation:

\[
\frac{\partial B}{\partial t} = \left( \frac{\eta}{R} \right) \frac{\partial}{\partial R} [R \frac{\partial B}{\partial R}]
\]

with solution

\[
B = \left( \frac{\phi_o}{4\pi\eta t} \right) \exp\left( -\frac{R^2}{4\eta t} \right)
\]

where \( \phi_o \) is the flux out to \( \infty \).

Sunspot flux to a certain stage

\[
F = \int_0^a 2\pi BR \, dR = F_o - 4\pi\eta B_s t
\]
Decay rate = -dF/dt = 4\pi \eta B_s = 1.2 \times 10^8 B_{\text{max}} \text{ m}^2/\text{s} \text{ which roughly agrees with observations.}

Special Topics

Running Penumbral Wave

Afvén wave propagating along field in penumbra

Evershed Flow \text{ siphon effect}

Flows in Penumbra :

u: umbra, p: penumbra,

Inner part of penumbra has inflow, while the outer part has an outflow.
Classification of sunspots

α: single  β: dipole  γ: some mixed  δ: opposite polarity umbrae share a common penumbra.

Umbral Dots

Formation of sunspots:

Tanaka proposed the emergence of twisted flux ropes. Discussion of a recent paper by Ishi, Karokawa, and Takeuchi (1998). From the apparent motion of sunspots derive the 3-D structure of the flux ropes.
Fig. 4.—(a) Schematic summary of observed vortex-like motions of small umbrae surrounding F1. Pairs of small umbrae successively emerged at the leading edge of F1. At the east side of F1, small umbrae, mainly $p$-polarities, moved clockwise and formed P2. At the west and north sides of F1, small umbrae, all $f$-polarities, moved counterclockwise and formed F5 by merging with some parts of decaying F1. (b) A schematic model of
Homework

- Construct a two-dimensional sunspot model, assuming a potential field configuration, and 100% of filling factor. Compare the density, temperature and magnetic fields in the umbra and non-sunspot photosphere.