In the previous lecture (week 2), we have studied kinetics of individual particles to the extent that the field particles are described by a mean temperature and density (although maybe not Maxwellian) and the test particle is mono-energetic. The approach presents fundamental quantities such as equilibrium time scales and transport coefficients. However we have elsewhere noticed that we must turn into a kinetic theory of plasmas to incorporate the internal degrees of freedom of each particle species.

The previous lecture also illustrates that the complexity of plasma dynamics arises for several reasons: (1) the inter-particle interaction is so weak in a plasma and the mean free path is much longer than macroscopic inhomogeneity scale. This can leads to times for their plasma momentum distribution for equilibrium longer. (2) Since Debye length is larger, collective excitation of internal electromagnetic field is common. As a result conglomerates of charged particles could be coupled to each other electro-magnetically in a way they collectively behave, like a single dynamical entity–a *plasmon*. (3) A variety of different kinds of plasmons can exist, each with its own peculiar structure and dynamics. (4) Interaction of the particles with a magnetic fields adds more complexity.

In this lecture we want to get some flavor of (2)-(3), using a simplest examples. Although a kinetic approach is our goal, it is instructive to start with a much simpler approach, the cold plasma approach which can be valid at low frequencies where ions and electrons are locked together by electrostatic forces-like electrically conducting fluid and we can ignore their temperatures (Sect. 1) . In Section 2, we will look at some generalizations of the cold plasma theory to include electron pressure, or wave damping. At higher frequencies complex dynamics is supported by momentum space anisotropies and can be analyzed using a variant of the kinetic theory, Vlasov equation. We study the Vlasov equation and and check the limits of validity of the fluid theory in Section 3. The Vlasov theory is a linear theory. However the complexities and long mean free path of plasma produce nonlinear phenomena. We deal with them in Sect. 5 and 6.

Part 2. Kinetic Theory of Warm Plasmas

1. The Cold Plasma Approach

We start with the cold plasma theory although our ultimate goal is to study a kinetic theory of plasmas to understand the afore-mentioned effects. The cold plasma theory deals with linear waves propagating in a plasma ignoring the effects of the thermal motions of the particles. It is not only instructive, but some physical arguments based on cold plasma assumption along with its generalizations can guide us to the correct dispersion relation for two simple modes in unmagnetized plasma: Langmuir waves and ion acoustic waves.

The first task is to understand propagation of linearized wave modes in uniform plasma. Write Maxwell's equation

$$\nabla\cdot\mathbf{E}=\frac{\rho_e}{\varepsilon_0},\qquad \nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad \nabla\times\mathbf{B}=-\mu_0\mathbf{j}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

¿From the last two equations,

$$\nabla\times(\nabla\times E)+\frac{1}{c^2}\frac{\partial^2E}{\partial t^2}=-\mu_0\frac{\partial j}{\partial t}$$

Fourier transforming

$$\mathbf{k}\times[\mathbf{k}\times\mathbf{E}(\omega,\mathbf{k})]+\frac{\omega^2}{c^2}\mathbf{E}(\omega,\mathbf{k})=i\mu_0\omega j(\omega,\mathbf{k})$$

or

$$\Big[\frac{c^2}{\omega^2}k_ik_j-\frac{c^2}{\omega^2}k^2\delta_{ij}+\delta_{ij}+\frac{i}{\varepsilon_0\omega}\sigma_{ij}\Big]E_j=0$$

where we used $J_i = \sigma_{ij} E_j$ and $\mu_0 \epsilon_0 = 1/c^2$. Alternatively we write

$$\begin{aligned} L_{ij} E_j &= 0, \\ L_{ij} &= \frac{c^2}{\omega^2}(k_i k_j - k^2 \delta_{ij}) + \epsilon_{ij} \\ \epsilon_{ij} &= \delta_{ij} + \frac{i}{\varepsilon_0 \omega} \sigma_{ij}(\omega, \mathbf{k}) \end{aligned} \tag{1}$$

The quantity ϵ_{ij} is called *dielectric tensor.* ϵ_0 is the dielectric permittivity of the vacuum. Specifying ϵ_{ij} may be regarded completely specifying the (linear) electromagnetic properties of a medium. Thus by saying "thermal unmagnetized" plasma, one means that $\epsilon_{ij}(\omega, \mathbf{k})$ is taken to be the expression obtained under the assumption that the plasma is thermal and unmagnetized.

For (1) to have a solution, the determinant of L should vanish, $|L_{ij}| = 0$. This gives a polynomial for the angular frequency $\omega(k)$, the solution of which gives a particular wave mode. In order to solve it, we must give a prescription for calculating the conductivity tensor σ_{ij}.

A simple prescription involves treating electrons and ions collectively as fluids. As shown in the previous lecture, collisions can be rare and we usually ignore them. The density n_s and mean velocity u_s of each species s must satisfy the equation of continuity and an equation of motion

$$\begin{aligned} \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) &= 0 \\ n_s m_s \Big[\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla) \mathbf{u}_s \Big] &= -\nabla \mathbf{P}_s + n_s q_s (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \end{aligned}$$

where $\mathbf{P}_s = n_s m_s \langle (\mathbf{v} - \mathbf{u}) \times (\mathbf{v} - \mathbf{u}) \rangle$, the pressure tensor.

By a "cold" plasma we mean $P = 0$, and further simplify considering an unmagnetized plasma ($B_0 = 0$) and only keep the linear wave modes,

$$\begin{aligned} -i\omega n_s m_s u_s &= q_s n_s E \\ j = \sum_s i n_s q_s u_s &= \sum_s \frac{i n_s q_s^2}{m_s \omega} E \end{aligned}$$

We then obtain the linearized current density and the conductivity tensor:

$$\mathbf{j} = \sum_s i n_s q_s \mathbf{u}_s = \sum_s \frac{i n_s q_s^2}{m_s \omega} \mathbf{E}, \qquad \sigma = \frac{i n_s q_s^2}{m_s \omega}.$$

Note that σ is pure imaginary to guarantee $\langle \mathbf{j} \cdot \mathbf{E} \rangle = 0$. Since σ is known we can express the dielectric tensor as

$$\epsilon_{ij} = \delta_{ij} + \frac{i}{\varepsilon_0 \omega} \sigma_{ij} = \Big(1 - \frac{\omega_p^2}{\omega^2}\Big) \delta_{ij} \qquad \text{with} \qquad \omega_p^2 = \sum_s \frac{n_s q_s^2}{m_s \epsilon_0}.$$

If we let $\mathbf{k} = k e_3$

$$\mathbf{L} = \begin{pmatrix} 1 - \frac{c^2 k^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} & 0 & 0 \\ 0 & 1 - \frac{c^2 k^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{pmatrix},$$

and use $||L_{ij}|| = 0$ to give the dispersion relation

$$\Big(1 - \frac{c^2 k^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}\Big)^2 \Big(1 - \frac{\omega_p^2}{\omega^2}\Big) = 0$$

Formally there are six solutions corresponding to three pairs of waves. Two pairs of waves are degenerate with frequency

$$\omega = \omega_p^2 + c^2 k^2$$

The associated eigenvectors have electric vectors lying along $\hat{e}_1$ and $\hat{e}_2$, i.e. perpendicular to $\hat{k}$. They are known as *electromagnetic modes.* We can use the 3rd Maxwell equation to solve for the magnetic perturbation

$$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega}$$

Note $\mathbf{B} \to 0$ as $\omega \to \omega_p$.

The other pair of modes only propagates with a single frequency, the plasma frequency in this limit.

$$1 - \omega_p^2/\omega^2 = 0$$

This is just the plasma oscillation (Langmuir waves) in which the whole plasma oscillates in phase.

Note also that

$$V_\phi = \frac{\omega}{k} = c\Big(1 - \frac{\omega_p^2}{\omega^2}\Big)^{-1/2} \geq c$$
$$V_g = \frac{\partial \omega}{\partial k} = c\Big(1 - \frac{\omega_p^2}{\omega^2}\Big)^{1/2} \leq c$$

Other important terms in wave propagation in cold plasma:
- Cut-off
- Resonance
- Faraday rotation
- Appleton-Hartree dispersion relation

2. Some Generalization of Cold Plasma Theory

So far we have completely ignored the effects of the thermal motions of the particles which can create pressure gradient and additional acceleration. Also wave damping or growth due to energy exchange between waves and particles. Here are some approaches without invoking kinetic theory.

Electron Thermal Pressure: In the above example, we completely ignored the effects of the thermal motions of the particles. As a result we obtained just the plasma oscillation (Langmuir waves) in which the whole plasma oscillates in phase. inclusion of electron pressure however makes the mode more dispersive. To understand this effects we still use fluid equation but keep the electron pressure nonzero.

$$\delta n_e = n_e \frac{\mathbf{k} \cdot \mathbf{u}}{\omega}, \quad \delta p_e = m_e s^2 \delta n_e$$

where n_e is the equilibrium electron density and $s = \sqrt{3kT_e/m_e}$.

This leads to Böhm-Gross dispersion relation:

$$\omega^2 = \omega_{P_e}^2 + \frac{3kT_e k^2}{m_e} = \omega_{P_e}^2 (1 + \lambda_D^2 k^2)$$

Now the dispersion relation has a finite width in frequency associated with the internal degree of freedom - thermal motion.

Another thing that we should however consider is the possible wave damping associated with energy exchange between waves and particles. Think about the phase velocity of Langmuir waves

$$V_\psi = \Big(\frac{kT_e}{m_e}\Big)^{1/2} \Big(3 + \frac{1}{k^2 \lambda_D^2}\Big)^{1/2}$$

When $\lambda < \lambda_D$, $V_\phi \sim V_{e,th}$, the phase speed becomes comparable with the electron thermal speed. It is then possible for individual electrons to communicate energy between adjacent compression &

rarefactions in the wave. The waves will then be damped and the above dispersion relation is no longer valid. In other words the Bohm-Gross dispersion relation is only valid for $\lambda \geq \lambda_D$.

Collisional Damping: As implied in the above, study of the wave damping process requires a kinetic approach. However, we take a look at such a process in a fluid approximation. Consider a transverse e.m. mode propagating in an unmagnetized partially ionized gas in which the electron-neutral collision frequency is ν_c. We introduce a term $-n_e m_e \nu_c \mathbf{u}$ to the equation of motion.

$$-i\omega n_e m_e u_e = q_e n_e E - n_e m_e \nu_c u_e$$
$$\text{so,} \qquad u_e = q_e n_e E/(\nu_c - i\omega) n_e m_e$$

Ignoring ion motion, we put the above to the current density to get conductivity and the dielectric constant.

$$J = n_e q_e u_e = \frac{n_e e^2}{m_e(\nu_c - i\omega)} E = \frac{\epsilon_0 \omega_p^2}{\nu_c - i\omega} E$$
$$\epsilon_{ij} = \delta_{ij} + \frac{i}{\epsilon_0 \omega}(\kappa_e)_{ij} = \Big[1 - \frac{\omega_p^2}{\omega(\omega + i\nu_c)}\Big]\delta_{ij}$$

Choose coordinates such that $k = ke_3$ then

$$L_{ij} = \begin{pmatrix} -k^2 + (\omega^2/c^2)A & 0 & 0 \\ 0 & -k^2 + (\omega^2/c^2)A & 0 \\ 0 & 0 & (\omega^2/c^2)A \end{pmatrix}.$$

where $A = 1 - \omega_p^2/\omega(\omega + i\nu_c)$.

The two degenerate transverse modes

$$0 = -k^2 + \frac{\omega^2}{c^2}\Big[1 - \frac{\omega_p^2}{\omega(\omega + i\nu_c)}\Big]$$

This gives

$$c^2 k^2 = \omega^2 - \omega_p^2 + i\frac{\omega_p^2 \nu_c}{\omega} + O\Big(\frac{\nu_c}{\omega}\Big)^2$$

$$\omega^2 = \omega_p^2 + c^2 k^2 - i\frac{\omega_p^2 \nu_c}{\sqrt{\omega_p^2 + c^2 k^2}}$$

where $\nu_c \ll \omega$.

Note: the wave energy lost by damping should be balanced by the energy gain of the plasma via Ohmic heating. This can be shown explicitly by calculating the rate of energy per unit volume $-\nabla \cdot \mathbf{S}$ and the ohmic heating of the plasma $\mathbf{j} \cdot \mathbf{E}$.

Beam instability in Cold Plasma: Recall the dispersion relation for longitudinal plasma oscillation.

$$1 - \frac{\omega_p^2}{\omega^2} = 0$$

If we consider a simple cold plasma of electrons and ions at rest, this corresponds to the dispersion relation for Langmuir waves with $\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2$.

In some other frames in which electrons and ions are moving with speed u, the observed frequency is Doppler-shifted and so

$$\frac{\omega_p^2}{(\omega - ku)^2} = 1 \qquad (2)$$

where ω is now the measured in this frame. To generalize to several cold plasma streams, each moving with speed u_i,

$$1 - \frac{\omega_{p1}^2}{(\omega - ku_1)^2} - \frac{\omega_{p2}^2}{(\omega - ku_2)^2} = 0$$

The maximum value for the growth rate is

$$\omega_i = \frac{3^{1/2}\alpha^{1/3}\omega_p}{2^{4/3}}$$

Use $\omega_p \sim 10^5$ s^{-1}, $\alpha \sim 10^{-3}$, $V \sim 10^4$ km s^{-1}. The wave should grow in a length of 30 km.

3. Vlasov equation - One Particle distribution function

The above shows how to generalize cold plasma theory to accommodate several distinct beams. A consequence is instability. Computing response of individual beams is an example of a Lagrangian approach. The robust method for developing the kinetic theory of warm plasma is an Eulerian approach in which we ask how many particles are found in a fixed volume of one particle phase space.

For a plasma with large λ_D, the two particle collisions (the e.m. fluctuations on length scales comparable with the inter-particle spacing) will be relatively unimportant, and collective effects asso. with the mean electromagnetic field will dominate the plasma response. In other words we can average over a volume that is small compared with a Debye sphere, but still large enough to contain many particles. Under these conditions, we can express the continuity of particles belonging to species s in one particle phase space as:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla \cdot (\mathbf{a} f_s) = 0$$

where

$$\mathbf{a} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Now $\mathbf{x}$, $\mathbf{v}$ are independent variables; $\mathbf{E}$, $\mathbf{B}$ are functions of $\mathbf{x}$, t but not of $\mathbf{v}$ and the term $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$, therefore

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla_{\mathbf{v}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0$$

Thus we rewrite it

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

This is the Vlasov equation for a collisionless particles species in a plasma. The electromagnetic field can be computed using Maxwell's equations from the charge and current densities, which are given in terms of the distribution function by

$$\rho_e = \sum_s q_s \int d\mathbf{v} f_s, \qquad \mathbf{j} = \sum_s \int d\mathbf{v}\mathbf{v} f_s$$

To relate the Vlasov equation to the fluid approach we take its moment defining

$$n_s = \int d\mathbf{v} f_s, \qquad \mathbf{u}_s = \frac{1}{n_s} \int d\mathbf{v}\mathbf{v} f_s$$

$$\mathbf{P} = m_s \int d\mathbf{v} (\mathbf{v} - \mathbf{u}) \times (\mathbf{v} - \mathbf{u}) f_s$$

Integrating the Vlasov equation over velocity space,

$$\frac{\partial n_s}{\partial t} + \frac{\partial (n_s \mathbf{u}_s)}{\partial \mathbf{x}} = 0.$$

To solve for n_s, we need to know $\mathbf{v}$. We multiply the Vlasov equation by $\mathbf{v}$and integrate over velocity space

$$\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s = \frac{1}{n_s m_s}[-\nabla \cdot \mathbf{P}_s + (\rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B})].$$

To solve for n_s and $\mathbf{u}_s$ we need to know $\mathbf{P}_s$. In order to know $\mathbf{P}_s$, we multiply the Vlasov equation by $\mathbf{v} \times \mathbf{v}$ and integrate over the third moment $\int d\mathbf{v}\ \mathbf{v} \times \mathbf{v} \times \mathbf{v}$, related to heat flux. This procedure will never terminate unless we introduce a closure relation. Typically we choose either the heat flux to vanish or assume $P_s = n_s k T_s$. Our fluid theory is certainly no more accurate than this closure relation.

4. Landau Damping

As a simple application, we use the Vlasov equation to re-derive the dispersion relation for Langmuir waves in an unmagnetized plasma. Suppose the ions are at rest, and electrons move in one dimension to consider only 1-D distribution function for electrons, $f(x, v, t)$. Also consider small amplitude waves so that we can use a linear treatment.

$$f(x, v, t) = f_0(v) + f_1(x, v, t) + ...$$

where $f_0(v)$ is the distribution function of the unperturbed electrons in the absence of waves and f_1 is the induced. If we linearize the Vlasov equation,

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = \frac{eE}{m_e}\frac{\partial f_0}{\partial v}.$$

Now seek a wave solution with f_1, $E \propto \exp[i(kx - \omega t)]$. We can solve for f_1

$$f_1 = \frac{ieE}{m_e(\omega - kv)}\frac{df_0}{dv}.$$

The system can be closed using Poisson's equation:

$$\frac{\partial E}{\partial x} = ikE = -\frac{e}{\epsilon_0}\int dv\ f_1.$$

Substitute for f_1 from (4.3) we obtain the dispersion relation, which we can express as the vanishing of the longitudinal dielectric coefficient $\epsilon_3 = \epsilon$ hence

$$\epsilon(k, \omega) = 1 + \frac{e^2}{m_e \epsilon_0 k}\int \frac{dv\ f_0'}{\omega - kv} = 0 \tag{3}$$

where $f_0' = df_0/dv$. For now on we suppress 0 in the unperturbed distribution function.

Note we can see its connection to the previous wave dispersion relation. To do so, perform the integral by part and rewrite the dispersion relation as

$$\frac{e^2}{m_e \epsilon_0}\int \frac{f}{(\omega - kv)^2}dv = 1.$$

This is the natural generalization of the two stream equation (Eq [2]).

How do we perform the integral in (3)? In a proper approach, we should set up an initial value problem and take a Laplace transform in time to solve for the temporal evolution of the field. However, if we're not concerned with the transient and we can just persist with Fourier analysis in time supplemented by a fairly subtle though important physical argument.

Consider a mode with $\omega = \omega_r + i\omega_i$. When $\omega_i > 0$ the wave will grow exponentially and the initial disturbance is small in the distant past and so it does not influence the present much. The calculation of the dielectric constant does not include the effect of the transients associated with the initial condition as it considers a particular frequency. Thus (3) can be just used in its form. If $\omega_i > 0$ the amplitude was large in the past, and we cannot assume that the transients are ignorable. It is this point that we are introducing causality into our time-symmetric equations by allowing the past (but not the future) to influence the present. The dielectric response $\epsilon(k, \omega)$ should be analytic in ω. Now formally allow v be complex and the integral should be performed in complex plane. When $\omega_i > 0$ the integral is along real v axis. However, if $\omega_i \leq 0$ the only the function can remain analytic is by deforming the contour away from the real axis so as to pass below the pole. Mathematically we use the analytic continuation of ϵ in the range for which our physical argument fails. This rule is known as the Landau prescription and the contour is called the Landau contour.

$$\int_L \frac{dv f'(v)}{v - \omega/k} = \int_{PV} \frac{dv f'(v)}{v - \omega/k} + i\pi f'(v = \omega/k)$$

where PV is the Cauchy principal value of the integral. If ω_i is small, we can Taylor expand away from this solution and invoke the Cauchy-Riemann equations to convert to derivatives along the real ω axis where we can evaluate the dielectric function.

$$\epsilon(k, \omega_r + i\omega_i) \approx \epsilon(k, \omega_r) + i\omega_i \frac{\partial \epsilon_r}{\partial \omega_r}$$

Having expanded the dielectric tensor we can evaluate the dispersion relation by making it vanish. From (3)

$$1 - \frac{e^2}{m_e \epsilon_0 k^2}\Big[\int_{PV} \frac{dv f'(v)}{v - \omega_r/k} + i\pi f'(\omega_r, k)\Big] - \frac{ie^2\omega_i}{m_e\epsilon_0 k^2}\frac{\partial}{\partial \omega_r}\int_{PV} \frac{dv f'(v)}{v - \omega_r/k} = 0 \tag{4}$$

For later use, we work out the imaginary part of (4) here

$$-\frac{\pi e^2}{m_e \varepsilon_0 k^2} f' = \frac{e^2 \omega_i}{m_e \varepsilon k^2}\frac{\partial}{\partial \omega_r}\int_{PV} \frac{dv f'(v)}{v - \omega_r/k} = -\omega_i \frac{\partial \epsilon_r}{\partial \omega_r}$$

Hence

$$\frac{\pi e^2}{m_e \varepsilon_0 k^2} f' \simeq k^2 \omega_i \frac{\partial \epsilon_r}{\partial \omega_r} \simeq 2k^2 \frac{\omega_i}{\omega_r} \tag{$*$}$$

Now to illustrate, let us determine the dispersion relation for a Maxwellian distribution function.

$$f = n\Big(\frac{m_e}{2\pi T}\Big)^{1/2} e^{-mv^2/2kT} \tag{5}$$

Our formalism is valid only $|\omega_i| \ll \omega_r$. In other words only a small fraction of the particles can resonate with the wave, which in turn requires $\omega_r/k \gg \sqrt{T/e}$. Evaluating the Cauchy Principal value,

$$
\begin{aligned}
\int_{PV} \frac{dv f'(v)}{v-\omega/k} &= \int_{-\infty}^{\infty} dv f' \Big[\frac{k}{\omega_r} + \frac{k^2}{\omega_r^2} v + \frac{k^3}{\omega_r^3} v^2 + \frac{k^4}{\omega_r^4} v^3 + ...\Big] \\
&= -\frac{nk^2}{\omega_r^2} - \frac{3n\langle v^2\rangle k^4}{\omega_r^4} + ... \\
&= -\frac{nk^2}{\omega_r^2}\Big(1 + \frac{3Tk^2}{m_e\omega_r^2} + ...\Big) \\
&\simeq -\frac{nk^2}{\omega_r^2}(1 + 3k^2\lambda_D^2)
\end{aligned} \tag{6}
$$

when $k\lambda_D \ll 1$.

Substituting (5), (6) into (4) and evaluating real and imaginary parts we obtain

$$
\begin{aligned}
\omega_r &= \omega_p(1 + 3k^2\lambda_D^2)^{1/2} \\
\omega_i &= -\Big(\frac{\pi}{8}\Big)^{1/2} \frac{\omega_p}{k^3\lambda_D^3} e^{-(k^2\lambda_D^2+3)/2}
\end{aligned} \tag{7}
$$

We have therefore demonstrated that 1) The Bohm-Gross dispersion relation has been recovered for the real part of the frequency. 2) As we increase wavelength ($k \to 0$) the wave phase velocity will increase so that a decreasing number of electrons will be resonant with the wave and the damping rate will decrease. 3) The sign of ω_i appears to depend on the sign of $f'(v)$ at resonance. When the derivative is negative (as typical), fewer particles will be traveling faster than the wave and adding energy to it than traveling slower and extracting energy. Therefore the wave is damped. 4) When $\omega/k \sim$ the mean thermal speed, the damping rate will be so severe that this calculation is invalid. We will see the opposite case in Sect. 6.

5. Quasi-Linear Theory

In the above we show the wave damps when the phase space density ($f(v)$) of the resonant particles decreases with increasing speed. Here we pay attention to what happens in the opposite case, i.e. when the distribution function rises with speed and waves can grow as a consequence. We use a formalism called *quasi-linear* or *weak turbulence theory*, which allows us to follow the time evolution of the waves and the particles simultaneously. We then interpret the formalism in quantum mechanical viewpoint.

Start from the Vlasov equation

$$
\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m_e}\frac{\partial f}{\partial v} = 0
$$

We take spatial average $\langle\ \rangle$ of this eqn and write $f = f_0 + f_1$ where $f_0 = \langle f\rangle$ is unperturbed distribution function and $f_1$the perturbation linear in the wave amplitude. We have

$$
\frac{\partial f_0}{\partial t} = -\frac{e}{m_e}\frac{\partial}{\partial v}\langle E f_1\rangle \tag{8}
$$

Next take the spatial Fourier transform of f_1

$$
\tilde{f}_1 = \int_0^L dx\ e^{-ikx} f_1
$$

where $L \gg k^{-1}$ is some averaging length. By Parseval's theorem,

$$
\langle E f_1\rangle = \frac{1}{L}\int_0^L dx E f_1 = \int \frac{dk}{2\pi}\frac{\tilde{E}^*\tilde{f}_1}{L} = \frac{e}{m_e}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{EE^*}{L}\frac{i}{\omega - kv}\frac{\partial f_0}{\partial v} \tag{9}
$$

The product $\tilde{E}\tilde{E}^*$ is a fcn of k, and there will be specific phase relationships between different Fourier components. If we increase however the size of the averaging volume all the phases may average to zero. This is called *Random Phase Approximation.* Sometimes it is valid. Sometimes it isn't when there is systematic bunching of the particles in phase.

Invoking the *RPA*, we can use Parseval's theorem to identify the average electrical energy in the waves

$$\frac{\epsilon_0 E^2}{2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{EE^*}{2L}$$

Introduce the wave spectral energy density, $\mathcal{E}_k$, which includes the oscillatory kinetic energy in the electrons and the electrical energy. $\int \mathcal{E}_k dk =$ total wave energy and $\mathcal{E}_k(-k) = \mathcal{E}_k(k)$. Hence,

$$\mathcal{E}_k = \epsilon_0 \frac{\langle EE^* \rangle_{RPA}}{\pi L} \tag{10}$$

We have taken only the spatial FT, implicitly we keep $\mathcal{E}_k \propto e^{2\omega_i t}$.

Combining (8), (9), (10)

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \mathcal{D} \frac{\partial f_0}{\partial v}$$

where the velocity diffusion coefficient is given by

$$\mathcal{D}(v) = \frac{e^2}{2\epsilon_0 m_e^2} \int dk \mathcal{E}_k \frac{i}{\omega - kv}$$

Simplify it by that $\omega(k)$ satisfies the dispersion relation (3). Since $\epsilon(k, \omega) = \epsilon(-k, -\omega^*)$, deduce that $\omega(k) = -\omega^*(-k)$, and rewrite

$$\mathcal{D}(v) = \frac{e^2}{2\epsilon_0 m_e^2} \int_0^k dk \mathcal{E}_k \frac{\omega_i}{(\omega_r - kv)^2 + \omega_i^2}$$

Now when the wave is weakly damped, i.e. $\omega_i \ll \omega_r$, further simplify by splitting up the integral into non-resonant and resonant parts.

$$\mathcal{D}(v) \simeq \frac{e^2}{2\epsilon_0 m_e^2} \int_0^k dk \mathcal{E}_k \Big[\frac{\omega_i}{(\omega_r - kv)^2} + \pi \delta(\omega_r - kv) \Big] \tag{11}$$

The first term in RHS is the nonresonant part. This exists because when a wave grows or decays f must change to reflect the changing oscillatory kinetic energy. We do not need to include it in QLT and use only the resonant part.

Quantum Mechanical Interpretation Naturally we can (second) quantize the field of plasma waves into quanta usually called *plasmons* and describe their creation and annihilation using quantum mechanical transition probabilities.

First, determine the occupation number of plasmons. These are scalar waves of spin zero, only one possible polarization state and the number of possible modes in a given volume of one particle phase space is given by $d\mathbf{x}d\mathbf{k}/(2\pi)^3$. The energy of individual plasmon is $\hbar\omega_r$. Therefore the occupation number is

$$n = \frac{(2\pi)^3 \mathcal{E}_k}{\hbar \omega_r}$$

When there is a resonant interaction between the wave field and resonant electrons in the plasma, the plasmon occupation number will vary according to

$$\frac{\partial n}{\partial t} = 2\omega_i n \tag{12}$$

where the factor 2 arises as we are dealing with a quantity proportional to the square of the wave amplitude.

The fundamental process we are dealing with is creation (or annihilation) of a plasmon by an electron. The kinematics is easy when the plasmon is much less energetic than electrons. We conserve energy and momentum in the interaction using

$$\hbar\omega_r = \triangle(m_e v^2/2) \simeq \triangle p \cdot v = \hbar \mathbf{k}\cdot\mathbf{v}$$

We therefore recover the resonance condition directly from the conservation laws.

Let the emission probability per unit time for an electron to emit a Langmuir wave spontaneously into an element of k-space be written as $W(\mathbf{v},\mathbf{k})$. The probability for induced emission of a plasmon into this state is then larger by the occupation number of the state n. Further the principle of *detailed balance* tells that W is also the relevant transition probability for the inverse process of absorption of the plasmon in a transition between the same two electron momentum states. Therefore write down a master equation as

$$\frac{\partial n}{\partial t} = \int dv W(v,k)\Big[f(v)[1+n(k)] - f(v-\frac{\hbar k}{m_e})n(k)\Big]$$

The first term refers to the spontaneous emission process, the second, induced emission, and the final term, induced absorption.

The momentum of a plasmon is so much smaller than the electron momentum that

$$f(v-\hbar k/m_e) \simeq f(v) - (\hbar/m_e)(\mathbf{k}\cdot\nabla_{\mathbf{v}})f$$

The difference of the induced emission and absorption terms can therefore be written as

$$\Big(\frac{\partial n}{\partial t}\Big)_i \simeq \int dv W n \frac{\hbar}{m_e}(\mathbf{k}\cdot\nabla_{\mathbf{v}})f$$

Using (*) and (12), we identify

$$W = \frac{\pi e^2 \omega_r}{\varepsilon_0 k^2 \hbar}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v})$$

Note that $W \sim \hbar^{-1}$ and a very large number. The classical absorption coefficient is the difference between the quantum mechanical induced absorption and emission rates. Under a normal condition $\partial f/\partial v < 0$ the absorption coefficient is positive, describing Landau damping. However when $\partial f/\partial v > 0$ there can be wave growth.

The spontaneous emission rate can be written

$$\Big(\frac{\partial n}{\partial t}\Big)_s = \int dv W(v,k) f(v)$$

This is *Cerenkov radiation*. Its associated absorption process is Landau damping. Substituting for W we have

$$\Big(\frac{\partial \mathcal{E}_k}{\partial t}\Big)_s = \frac{e^2}{8\pi^2\epsilon_0}\int dv f(v)\frac{\omega_r^2}{k^2}\delta(\omega_r - k\cdot v)$$

$$\begin{aligned}\frac{\partial f}{\partial t} = \int d\mathbf{k}(1+n)\Big[W(\mathbf{v}+\frac{\hbar\mathbf{k}}{m_e},\mathbf{k})f(\mathbf{v}+\frac{\hbar\mathbf{k}}{m_e}) - W(\mathbf{v},\mathbf{k})f(\mathbf{v})\Big]\\ - n\Big[W(\mathbf{v}+\frac{\hbar\mathbf{k}}{m_e},\mathbf{k})f(\mathbf{v}) - W(\mathbf{v},\mathbf{k})f(\mathbf{v})\Big]\end{aligned}$$

Expanding the electron distribution function to 2nd order and retaining those terms independent of $\hbar$ we obtain the quasi-linear electron kinetic equation

$$\frac{\partial f}{\partial t} = \nabla_{\mathbf{v}}\cdot[\mathbf{A}(\mathbf{v})f + \mathcal{D}\cdot\nabla_{\mathbf{v}} f] \tag{13}$$

where

$$\mathbf{A}(\mathbf{v}) = \int d\mathbf{k}\frac{W(\mathbf{v},\mathbf{k})\hbar\mathbf{k}}{m_e}$$
$$\mathcal{D} = \int d\mathbf{k}\frac{W(\mathbf{v},\mathbf{k})\hbar^2\mathbf{k}\times\mathbf{k}}{2m_e^2}$$

We can identify (13) in Fokker-Planck form. $\mathbf{A}$ is a resistive Fokker-Planck coefficient associated with spontaneous emission. $\mathcal{D} = \frac{1}{2}\langle\triangle\mathbf{v}\times\triangle\mathbf{v}/\triangle t\rangle$ is the combined resistive-diffusive coefficient that arises when electron recoil can be ignored and associated with induced processes.

6. Instabilities and Nonlinear effects

For illustration of the use of quasi-linear theory, we first show the Bump-in-tail instability in which a "warm" electron distribution function evolves so as to shut down the growth of the waves. Second, we show trapping of particles in a potential well set by the large amplitude waves.

Bump-in Tail Instability-isotropization of galactic cosmic rays: When a weak beam of electrons passes through a stable Maxwellian plasma with speed v_b large compared with the thermal width of the background plasma σ_e, this distribution is known as "bump in tail" distribution.

Let's approximate the beam by a Maxwellian

$$f_b(v) = \frac{n_b}{\sqrt{2\pi}\sigma_b}e^{-(v-v_b)^2/2\sigma^2}$$

where n_b is the beam electron density.

Suppose that the beam is established at $t = 0$ and the Langmuir wave energy density $\mathcal{E}_k$ is small.The wave will grow fastest when the slop of f is most positive, i.e. when $v = v_b - \sigma_e$. The associated max. growth rate is given by (12)

$$\omega_{i,max} = \left(\frac{\pi}{8e}\right)^{1/2}\left(\frac{v_b}{\sigma_b}\right)^2\left(\frac{n_b}{n_e}\right)\omega_p \tag{14}$$

Now modes will grow over a range of wave phase velocities $\triangle V_\phi \sim \sigma_b$. If we use the Bohm-Gross dispersion relation,

$$\omega = \omega_p(1 - 3\sigma_e^2/V_\psi^2)^{-1/2}$$

then we find that the range bandwidth of the growing modes is roughly

$$\triangle\omega = K\omega_p\frac{\sigma_b}{v_b} \tag{15}$$

where $K = 3(\sigma_e/v_b)^2[1 - 3(\sigma_e/v_b)^2]^{-3/2}$ is a constant ≥ 0.1 typically. Combining (14) and (15) we obtain

$$\frac{\omega_{i,max}}{\triangle\omega} \sim \left(\frac{\pi}{8eK^2}\right)^{1/2}\left(\frac{v_b}{\sigma_b}\right)^3\left(\frac{n_b}{n_e}\right)$$

Dropping constants of order unity, we conclude that wave growth time $\sim 1/\omega_{i,max}$ is long enough compared with the coherence time $\sim 1/(\triangle\omega)$ provided that

$$\sigma_b \geq \left(\frac{n_b}{n_e}\right)^{1/3} v_b \tag{16}$$

When this inequality is satisfied the wave will take several times to grow and so we expect that no permanent phase relation will be established in the electric field and the quasi-linear theory is appropriate. However, this inequality is reversed, the instability resembles more the two stream instability and the quasilinear theory is invalid.

With this restriction in mind, we associate a wave energy density not just with a given value of k but with a given $v = \omega/k$. Using (11) for the velocity diffusion coefficient and (*) for the associated wave growth rate, write

$$\frac{\partial f}{\partial t} = \frac{\pi e^2}{\epsilon_0 m_e \omega_p} v^2 \mathcal{E}_k(v) \frac{\partial f}{\partial v}$$

$$\frac{\partial \mathcal{E}_k}{\partial t} = \frac{\pi e^2}{m_e^2 \epsilon_0} \frac{\partial}{\partial v} \frac{\mathcal{E}_k}{v} \frac{\partial f}{\partial v}$$

See that a velocity space irregularity leads to the growth of electrostatic waves which can react back on the particles in such a way to saturate the instability. The net result is a beam of particles gets broader as propagating through the plasma, and the waves will ultimately damp.

An example of this type instability is the isotropization of galactic cosmic rays. The cosmic rays propagating through interstellar medium are effectively scattered by hydromagnetic Alfvén waves. It turns out that the particles are unstable to the growth of Alfvén waves satisfying the resonance condition: $\omega - k_\parallel \cdot v_\parallel = \omega_G/\gamma$. The growth rate of these waves can be studied using a kinetic theory similar to the above and is approximately given by $\omega_i \simeq (n_c r/n_e)\omega_{Gi}(u_{cr}/a - 1)$. The waves will then react back on the cosmic rays, resulting in isotropization of their momenta.

Nonlinear Landau Damping: A new physical phenomenon appears when inequality (16) is not satisfied and the growing wave turbulence is better described by a single coherent mode than by an incoherent superposition of random waves. If we look at wave in a frame moving with its phase velocity, then we see an oscillatory electric potential, and particle very close to resonance with the wave (i.e. with speed $\sim \omega/k$) will have very little energy in measured in the moving frame and therefore become trapped in the potential well created by the electrostatic potential. When this happens, the particles are said to be trapped. To approximate the shape of the potential well as harmonic, we write

$$\ddot{x} \simeq -\frac{eE \sin kx}{m_e} \simeq -\omega_b^2 x$$

where

$$\omega_b = \left(\frac{eEk}{m_e}\right)^{1/2}$$

is known as the bounce frequency. As the potential well is, in practice, anharmonic, these trapped particles will mix in phase rapidly. Once a particle is trapped its orbit is closed, and it is no longer capable of supplying the free energy to fuel the wave growth. Wave grows in a time scale of $\sim 1/\omega_i$. Wave trapping should be only important when the bounce time $\sim 1/\omega_b$ is short compared with this growth time.

Electron trapping causes particles to be bunched together at certain preferred phases of a growing wave. This can have important consequences for the radiative properties of the plasma. Suppose, for example, electrons are in orbit in a magnetostatic field. they will radiate *cyclotron radiation*. If their gyrational phases are random then the total power that they radiate will be sum of their individual particle powers. However if N electrons were localized at the same gyrational phase, they would then radiate like one giant electron with a charge Ne. As the radiated power is proportional to the square of the charges carried by the radiating particles. Now a mono-energetic distribution of gyrating mildly relativistic electrons can be unstable to the growth of cyclotron waves through this type of bunching. This is called *cyclotron masers*. Unstable waves will grow and the wave emission will increase until trapping of the particles leads to saturation of the growth of the wave. Cyclotron masers can be made in the laboratory and occur in nature for example, in the Jovian magnetosphere and in solar flares.

`Homework:  Complete the derivation of the quasi-linear equation (13).`